Supplemental Notes

EE503 Week 12 Dr. Franzke

HW #12

① Handout - financial engineering

Financial Engineering

Topics

① Random RAP (rational asset pricing)

$$E[P] = \frac{D_0(1+pg)}{r-pg} \quad \text{if } pg < r$$

present value P

probability p
growth g
discount rate r

Dividend Discount Model

② Random Walk is a martingale

$$X_k = X_{k-1} + N_k$$ noise

③ Martingale Pricing Theorem

"Can't beat an efficient market"

Compound Interest

Future Value = $B(1+r)^n$

B: Balance
n: years (time)
r: rate of return

∴ "Rule of 72"

- Doubling time n for rate of return r:

$$\boxed{n = \frac{0.72}{r}}$$

because $\cancel{B}(1+r)^n = 2\cancel{B}$ if $B > 0$

$$\therefore \; n = \frac{\ln 2}{\ln(1+r)}$$

$$= \frac{0.693}{r - \frac{r^2}{2} + \frac{r^3}{3} - \dots}$$

$$\approx \frac{0.72}{r}$$ since $\ln(1+r) < r$

∴ $r = 0.02$ return takes about 36 years to double

$r = 0.05$ takes about 14.5 years to double

$r = 0.10$ takes about 7 years to double

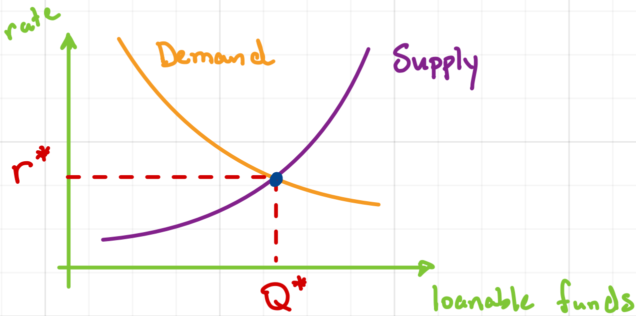
Discount factor : $\dfrac{1}{(1+r)^n}$

Ex: Inflation erosion of 3% per year

$\therefore n = 5$: $\dfrac{\$1}{(1+0.03)^5} \approx \$0.86$

$n = 30$: $\dfrac{\$1}{(1+0.03)^{30}} \approx \$0.412$

Interest rate r as cost of money :



Equilibrium : Supply = Demand

Bond present value P:

no uncertainty
(no default risk)

$$P = \underbrace{\sum_{k=1}^{n} \frac{C}{(1+r)^k}}_{\text{discounted cash flow}} + \underbrace{\frac{M}{(1+r)^n}}_{\text{discounted principal}}$$

C: semi-annual Coupon payment

P: (rational) Price of the bond

r: rate of return (half of required annual yield)

k: payment time period

M: bond's Maturity value ("par")

Proposition: $P = C \, \dfrac{1 - \frac{1}{(1+r)^n}}{r} + \dfrac{M}{(1+r)^n}$

Pf: $\displaystyle\sum_{k=1}^{n} \frac{1}{(1+r)^k} = \sum_{k=1}^{n} \Big(\underbrace{\frac{1}{1+r}}_{a} \Big)^k$ $(r > 0)$

$\displaystyle\sum_{k=1}^{n} a^k = \frac{a - a^{n+1}}{1-a}$ if $a \neq 0$ $\longrightarrow$

$$= \frac{\frac{1}{1+r} - \frac{1}{(1+r)^{n+1}}}{1 - \frac{1}{1+r}}$$

since finite sum

$$= \frac{\frac{1+r}{1+r} - \frac{(1+r)^1}{(1+r)^{n+1}}}{r}$$

$$= \frac{1 - \frac{1}{(1+r)^n}}{r}$$

QED

Corollary: $P = M$ if $C = r \cdot M$

price or "par"

Pf: $P \overset{\text{prop}}{=} C \, \frac{1 - \frac{1}{(1+r)^n}}{r} + \frac{M}{(1+r)^n}$

$$\overset{\text{hyp.}}{=} r \cdot M \, \frac{1 - \frac{1}{(1+r)^n}}{r} + \frac{M}{(1+r)^n}$$

$$= M \left(1 - \cancel{\frac{1}{(1+r)^n}} + \cancel{\frac{1}{(1+r)^n}}\right)$$

$$= M$$

QED

US Treasuries

T-bills: Maturity ≤ 1 year (often in weeks)

T-notes: 2, 3, 5, 7, and 10 years

- interest paid "semi-annually"

interest rate proxy: yield on 10 year note

T-bond: 30 year maturity

See: $\boxed{\text{Treasury Direct}}$ ☆

- Buy / sell bonds directly with Federal Reserve
- You bank maintains account (at no charge)

Need summation result for truncated geometric series

Thm: $\displaystyle\sum_{j=k}^{\infty} a^j = \frac{a^k}{1-a}$ if $|a| < 1$ for $k = 0, 1, 2, \ldots$

Pf: put $m = j-k$ in finite n-sum $\displaystyle\sum_{j=k}^{n} a^j$ if $j \le n$

$\therefore\ j = m+k$

$$\therefore \sum_{j=k}^{j=n} a^j \overset{m=j-k}{=} \sum_{m+\cancel{k}=\cancel{k}}^{m+k=n} a^{k+m} = \sum_{m=0}^{m=n-k} a^{m+k}$$

$$= a^k \sum_{m=0}^{n-k} a^m$$

$$= a^k \left(\frac{1-a^{n-k+1}}{1-a} \right)$$ since $a \ne 1$ and $n < \infty$

$$\therefore \sum_{j=k}^{\infty} a^j = \lim_{n\to\infty} \sum_{j=k}^{n} a^j = \lim_{n\to\infty} a^k \left(\frac{1-a^{n-k+1}}{1-a} \right)$$

$$= a^k \left(\frac{1-\lim_{n\to\infty} a^{n-k+1}}{1-a} \right) = a^k \left(\frac{1-0}{1-a} \right)$$ since $|a| < 1$ and defn of limit

$$= \frac{a^k}{1-a}$$

QED

Corollary: $\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$ if $|a|<1$

Rational Asset Pricing (RAP)

Idea: Present value = Discounted future cash flow

Dividends: $D_0, D_1, D_2, \ldots$ ← cash flow

D_0 = current dividend (profit)

Constant Growth g:

$$D_1 = D_0 + g \cdot D_0 = D_0(1+g)$$

$$\therefore D_n = D_0(1+g)^n$$

"We define intrinsic value as the discounted value of the cash that can be taken out of a business during its remaining life. Anyone calculating intrinsic value necessarily comes up with a highly subjective figure that will change both as estimates of future cash flows [*g*] are revised and as interest rates [*r*] move. Despite its fuzziness, however, intrinsic value is all-important and is the only logical way to evaluate the relative attractiveness of investments and businesses."

Warren E. Buffet

Annual Report of Berkshire Hathaway: 1994

RAP Theorem

Discounted Discount Model (1-stage)

(Gordon-Shapiro, 1956
John Burr Williams, 1938)

$$\boxed{P = D_0 \frac{1+g}{r-g}}$$

memorize (and derive)

if $g < r$

D_k = Dividend at future time k

g = Constant Growth Rate of cash flow $D_0, D_1, D_2, \ldots$

r = Discount rate (often estimate with CAP model)

P = present value of asset

Pf: $$P = \sum_{n=1}^{\infty} \frac{D_n}{(1+r)^n}$$ discounted dividend flow

$$= \sum_{n=1}^{\infty} \frac{D_0(1+g)^n}{(1+r)^n}$$ constant growth g: $D_n = D_0(1+g)^n$

$$= D_0 \sum_{n=1}^{\infty} \frac{(1+g)^n}{(1+r)^n} = D_0 \sum_{n=1}^{\infty} \underbrace{\left(\frac{1+g}{1+r}\right)^n}_{a_n}$$

geometric series

kal, Corr.

$$= D_0 \frac{\frac{1+g}{1+r}}{1 - \frac{1+g}{1+r}}$$

since $g < r$ by hypo.

$\therefore 1+g < 1+r$

$\therefore a = \frac{1+g}{1+r} < 1$

$$= D_0 \frac{1+g}{1+r - \frac{\cancel{(1+r)}(1+g)}{\cancel{1+r}}}$$

$$= D_0 \frac{1+g}{\cancel{1}+r-\cancel{1}-g}$$

$$= D_0 \frac{1+g}{r-g}$$

QED

$\therefore$ <u>Corollary</u> 1: if $g < r$ $\quad S = \frac{D_0(1+g)}{O(r-g)}$

rational <u>Share Price</u>

where S = share price

O = # shares outstanding

with $S = \frac{P}{O}$

☆ <u>Corollary</u> 2: if $g < r$ $\quad r = \underbrace{\frac{D_0(1+g)}{OS}}_{\text{yield}} + \underbrace{g}_{\text{growth}}$

Can view r and g as random variables

Corollary 3: if $0 < r$ and if $g = 0$ (no growth)

$$\boxed{P = \frac{D}{r}}$$

$$\therefore \underbrace{r}_{\text{discount rate}} = \underbrace{\frac{D}{P}}_{\text{yield}}$$

Corollary 3 corresponds to an "infinite bond"

$$P^{prop} = \lim_{n \to \infty} \left(C \cdot \frac{1 - \frac{1}{(1+r)^n}}{r} + \frac{m}{(1+r)^n} \right)$$

$$= C \cdot \frac{1 - \lim_{n \to \infty} \frac{1}{(1+r)^n}}{r} + m \cdot \lim_{n \to \infty} \frac{1}{(1+r)^n}$$

$$= C \cdot \frac{1-0}{r} + 0 \qquad \text{since } r > 0$$

$$= \frac{C}{r} = \frac{D}{r} \qquad \text{if } C = D$$

i.e. coupon = dividend (fixed)

Time Horizon N

$\therefore$ **Thm:** 2-stage RAP (piecewise constant growth)

Suppose $\begin{cases} g_1 > r & \text{for } N \text{ years} \\ g_2 < r & \text{for } N+1, N+2, \ldots \end{cases}$

Then $$P = \underbrace{D_0 \left(\frac{a - a^{N+1}}{1-a} \right)}_{\text{stage 1}} + \underbrace{\frac{D_{N+1}}{r-g_2} \cdot \frac{1}{(1+r)^N}}_{\text{stage 2 "tail"}}$$

if $a = \dfrac{1+g_1}{1+r} > 1$

But: hard to estimate N

RAP Application

Coca-Cola (KO)

NYSE symbol

- World leading BRAND-name stock
- Warren Buffet (Berkshire) owns about 8% of KO stock

Θ = 2.3 Billion shares outstanding

- 139,600 employees
- Profit margin: 27.6% (very high)
- Revenue (2011): \$46 B
- Net income (profit): $\underbrace{\$12.7\text{ B}}_{D_0}$
- Share price ≈ \$68/share
- Beta $\beta = 0.42$ (very low)

  measure of volatility relative to S&P 500

Annual Report Data for Coca-Cola:

	2003	2004	2005	2006	2007	2008	2009	2010	(est.) 2011
Revenue (in billions)	20.9	21.7	23.1	24.1	28.9	31.9	30.8	35.1	46
Net income (profit)	4.3	4.8	4.9	5.1	5.9	5.8	6.8	11.8	12.7 = D_0

Roughly constant growth

Net income: $\pi: \; 4.3(1+g)^8 = 12.7$

$$\longleftrightarrow \quad g_\pi = \sqrt[8]{\frac{12.7}{4.3}} - 1 \approx 1.14 - 1 = \overset{\text{high}}{0.14}$$

Revenue: $20.9(1+g_R)^8 = 46$

$$\longleftrightarrow \quad g_R = \sqrt[8]{\frac{46}{20.9}} - 1 \approx 1.10 - 1 = 0.10$$

$\therefore$ top-line growth SLOWER than bottom line growth

Experiment with different discount rates r (and g values)

Case 1: Pick $r = 0.15$ $(g = 0.14)$

$$\therefore P = \frac{D_0 \cdot (1+g)}{r-g} = \frac{\$12.7 \times 1.14}{0.01} = \$1447.8B$$

$$\therefore S = \frac{P}{O} = \frac{\$1447.8\cancel{B}}{2.3\cancel{B}} = \$629.5/\text{share}$$

Case 2: $r = 0.2$ $\quad \therefore P = \frac{\$12.7 \times 1.14}{0.06} = \$241.3\,B$

$$\therefore S = \frac{P}{O} = \frac{\$241.3\,B}{2.3\,B} = \$104.9/\text{share}$$

Case 3: $r = 0.15$, but $g = 0.10$

$$\therefore P = \frac{D_0(1+g)}{r-g} = \frac{\$12.7 \times 1.1}{0.05} = \$279.4\,B$$

$$\therefore S = \frac{P}{O} = \frac{\$279.4\,B}{\$2.3\,B} = \$121.5/\text{share}$$

Case 4: "Market" discount rate

- solve for r

$$\therefore r = \frac{D_0(1+g)}{OS} + g$$

$$= \frac{\$12.7 \times 1.14}{2.3 \times \$68} + 0.14 = 0.0925 + 0.14$$

$$= 0.233$$

$$\therefore r_{market} = 23.3\%$$

Check: $S = \frac{P}{O} = \frac{D_0(1+g)}{O(r_m - g)} = \frac{\$12.7 \times 1.14}{2.3 \times 0.093} = \$67.7 / \text{share}$

current share price $68

$$\left(g = 0.1 \longrightarrow r_{market} \approx 18.9\% \right)$$

☆ **Thrm:** Random RAP (semi-constant growth)

If
- D_0 known (deterministic)
- $\underset{\text{random variable}}{D_{k+1}} = \begin{cases} D_k(1+g) & \text{w/ probability } p \\ D_k & \text{w/ } 1-p \end{cases}$
- $pg < r$

Then
$$\boxed{E[P] = D_0 \frac{1+pg}{r-pg}}$$
☆

∴ Deterministic RAP if $p=1$

Pf:

(A)
$$E[D_{k+1}] \overset{\text{total exp}}{=} E_{D_k}\big[E[D_{k+1} \mid D_k]\big]$$
$$\overset{\text{dividend hypo.}}{=} E_{D_k}\big[pD_k(1+g) + (1-p)D_k\big]$$
$$= E_{D_k}\big[\cancel{pD_k} + pgD_k + D_k - \cancel{pD_k}\big]$$
$$= E_{D_k}\big[D_k + pgD_k\big]$$
$$= E_{D_k}\big[(1+pg)D_k\big]$$
$$\overset{\text{linearity}}{=} (1+pg)^{1} E_{D_k}[D_k]$$
$$\overset{\text{total exp}}{=} (1+pg)\cdot E_{D_{k-1}}\big[E[D_k \mid D_{k-1}]\big]$$

$$\overset{\text{hyp.}}{=} (1+pg)\, E_{D_{k-1}}\left[pD_k(1+g) + (1-p)D_k\right]$$

$$= (1+pg)\, E_{D_{k-1}}\left[\cancel{pD_{k-1}} + pgD_{k-1} + D_{k-1} - \cancel{pD_{k-1}}\right]$$

$$= (1+pg)\, E_{D_{k-1}}\left[(1+pg)D_{k-1}\right]$$

$$\overset{\text{lin.}}{=} (1+pg)^2\, E_{D_{k-1}}\left[D_{k-1}\right]$$

$$= \cdots$$

$$\vdots$$

$$= (1+pg)^k \cdot E_{D_1}\left[D_1\right]$$

$$\overset{\text{hyp.}}{=} (1+pg)^k \cdot \left(pD_0(1+g) + (1-p)D_0\right)$$

$$= (1+pg)^k \left(\cancel{pD_0} + pgD_0 + D_0 - \cancel{pD_0}\right)$$

$$= (1+pg)^k \left((1+pg)D_0\right)$$

$$= (1+pg)^{k+1} \cdot D_0 \leftarrow \text{constant by hypothesis}$$

$$\therefore (A) \quad \boxed{E_{D_{k+1}}\left[D_{k+1}\right] = (1+pg)^{k+1} D_0}$$

$$(B) \quad E[P] = E\left[\sum_{k=1}^{\infty} \frac{D_k}{(1+r)^k}\right] \overset{\text{lin}}{=} \sum_{k=1}^{\infty} \frac{E[D_k]}{(1+r)^k}$$

$$\overset{(A)}{=} \sum_{k=1}^{\infty} \frac{D_0(1+pg)^k}{(1+r)^k} = D_0 \sum_{k=1}^{\infty} \frac{(1+pg)^k}{(1+r)^k} \quad \text{since } D_0 \text{ deterministic}$$

$$= D_0 \sum_{k=1}^{\infty} \left(\frac{1+pg}{1+r}\right)^k \quad \text{geometric sum}$$

$$= D_0 \frac{\left(\frac{1+pg}{1+r}\right)}{1-\left(\frac{1+pg}{1+r}\right)}$$

if $pg < r$

since then $\frac{1+pg}{1+r} < 1$

from Corollary $(k=1)$

$$= D_0 \frac{1+pg}{1+r-\left(\frac{1+pg}{\cancel{1+r}}\right)\cancel{(1+r)}}$$

$$= D_0 \cdot \frac{1+pg}{r-g}$$

QED

Similarly:

Thrm (1): If
- $g(p-p_D) < r$ & D_0 known (deterministic)
- $\begin{cases} D_K(1+g) & \text{with } p \\ D_K(1-g) & \text{with } p_D \text{ (probability of DECREASE)} \\ D_K & \text{with } 1-p-p_D \end{cases}$

Then

$$\boxed{E[P] = D_0 \frac{1+g(p-p_D)}{r-(p-p_D)g}}$$

Thrm: Bankruptcy RAP

(2)

If $-\ (p - p_D)\,g < r + p_B$ & D_0 deterministic

$$- \quad \begin{cases} D_k(1+g) & \text{with } p \\ D_k(1-g) & \text{with } p_D \\ D_k & \text{with } 1 - p - p_D - p_B \\ 0 & \text{with } p_B \end{cases}$$

(probability of BANKRUPTCY)

Then
$$\boxed{E[P] = D_0 \left(\frac{1 + (p - p_D)\,g - p_B}{r - (p - p_D)\,g + p_B} \right)}$$

Martingale Properties and the Martingale Pricing Theorem

Defn: r.v. $\{X_n\}$ is a Martingale process iff

$$\begin{cases} (1) \ \forall n: E[|X_n|] < \infty \\ (2) \ E[X_{n+1} \mid X_1, \dots, X_n] = X_n \end{cases}$$

"fair game"

with σ-algebra "Filtration" $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots$

s.t. $X_k \xleftrightarrow{\text{measurable}} \mathcal{Q}_k$

($\therefore$ $\{X_k\}$ "adapted" to filtration)

Sub-Martingale: $E[X_{n+1} \mid X_1, \dots, X_n] \geq X_n$

$\therefore$ casino is a submartingale

Super-Martingale: $E[X_{n+1} \mid X_1, \dots, X_n] \leq X_n$

$\therefore$ gambler is a supermartingale

Ex: If $\{X_n, \mathcal{Q}_n\}$ is a martingale

then $\{|X_n|, \mathcal{Q}_n\}$ is a submartingale

- so is $\{|X|^p\}$ if $1 < p < \infty$

and if $\forall n: E[|X_n|^p] < \infty$

Recall: $\mathcal{Q} \overset{CIA}{=} \sigma(S)$ if $S \subset 2^{\Omega}$

$\therefore \mathcal{Q}_n = \sigma(\{X_1, \ldots, X_n\})$

minimal S.A. for which $X_1, \ldots, X_n$ are measurable (PAM)

Claim: $Z_n = E[X | Y_1, \ldots, Y_n] = E[X | \mathcal{Q}_n]$

is a martingale if $\mathcal{Q}_n = \sigma(Y_1, \ldots, Y_n)$

Pf: $E[Z_{n+1} | Z_1, \ldots, Z_n] = E[Z_{n+1} | \mathcal{Q}_k]$ since $\mathcal{Q}_k \subset \mathcal{Q}_{k+1}$

$\overset{\text{def } Z}{=} E[E[X | \mathcal{Q}_{n+1}] | \mathcal{Q}_n]$

$\overset{\text{total exp}}{=} E[X | \mathcal{Q}_n]$

$= Z_n$

$\therefore$ martingale QED

Thrm: $S_n = \sum_{k=1}^{n} X_k$ is a <u>martingale</u> if

- $X_1, \dots, X_n$ are independent
- $E[|X_n|] < \infty \quad \forall n$
- $E[X_n] = 0 \quad \forall n$

Pf: (1)

$$E[|S_n|] = E[|X_1 + \cdots + X_n|]$$
$$\leq E[|X_1| + \cdots + |X_n|]$$
$$= \sum_{k=1}^{\infty} E[|X_k|] < \infty$$

since each $E[|X_k|] < \infty$ by hypothesis

(2)

$$E[S_{n+1} \mid S_1, \dots, S_n] = E[S_{n+1} \mid X_1, \dots, X_n]$$

since $\mathcal{F}_n = \sigma(S_1, \dots, S_n) = \sigma(X_1, \dots, X_n)$

$$\overset{\text{defs}}{=} E[S_n + X_{n+1} \mid X_1, \dots, X_n]$$
$$\overset{\text{lin}}{=} E[S_n \mid X_1, \dots, X_n] + E[X_{n+1} \mid X_1, \dots, X_n]$$
$$\overset{\text{ind}}{=} E[S_n \mid X_1, \dots, X_n] + E[X_{n+1}]$$
$$\overset{E[X_k]=0}{=} E[S_n \mid X_1, \dots, X_n] + 0$$
$$= E[f(X_1, \dots, X_n) \mid X_1, \dots, X_n]$$

since $S_n = \sum_{k=1}^{n} X_k = f(X_1, \dots, X_n)$

$$= f(X_1,\dots,X_n)\cdot E[1 \mid X_1,\dots,X_n]$$

since $E[g(X)Y|X] = g(X)\cdot E[Y|X]$

$$= f(X_1,\dots,X_n) = S_n$$

$\therefore \{S_n\}$ is a martingale

QED

Thm: Doob's Maximal Inequality

If
- $\{X_n\}$ is a SUBmartingale
- $X_n \geq 0 \quad \forall n$
- $c > 0$
- $X_n^* \equiv \sup_{0\leq m\leq n} X_n$ — "maximal sequence"

then $P(X_n^* \geq c) \leq \dfrac{E[X_n]}{c}$

generalizes Markov (and Kolmogorov) inequality

$\therefore$ ☆ **Thm:** L^2-Bounded Martingale Convergence Theorem ☆

If
- $\{X_n\}$ is a martingale
- $\{X_n\}$ is "L^2-bounded" — $\exists B : B<\infty \;\&\; E[X_n^2]\leq B \;\forall n$

then (1) $\exists$ r.v. X : $E[X^2] \leq B$

(2) $X_n \xrightarrow{\circ} X$

(3) $X_n \xrightarrow{m} X$

((2) still holds in "L^1" case of $E[|X_n|] \le B < \infty$)

$\therefore$ - "Martingale converges to something X"

- implies some form of SLLN

- Martingale CLT exists that allows correlated samples (but ergodic)

Application to Finance

Idea 1: S.A. $\mathcal{C}_n = I_n$ $\quad I_1 \subset I_2 \subset \cdots \subset I_n$

= Information available at n

Idea 2: "Rational Expectations"

Rational Prices $\approx E[\text{PRICES} \mid \text{all information } I_n]$

- John Muth (1950s)
- Robert Lucas (Nobel Prize: 1995)

- Can't beat a perfectly efficient market

$\therefore$ Show $\{p_t\}$ is a martingale

"Random Walk Down Wall Street"

– B. Malkiel

Ex: Random Walk as martingale

- $X_k = X_{k-1} + N_k$
- X_k and N_j independent
- Noise N_k : $Cov(N_s, N_t) = 0$ if $s \neq t$.
- $E[N_k] = 0$

$$\therefore E[X_{k+1} \mid X_1, \ldots, X_n] \overset{\text{def walk}}{=} E[X_k + N_{k+1} \mid X_1, \ldots, X_n]$$

$$= E[X_k \mid X_1, \ldots, X_n] + E[N_{k+1} \mid X_1, \ldots, X_n]$$

$$= X_k + E[N_{k+1} \mid X_1, \ldots, X_n]$$

since $X_k = g(X_1, \ldots, X_k)$ trivially

$$= X_k + E[N_{k+1}]$$

since $Cov(N_{k+1}, N_s) = 0$ & X_k and N_j independent

$$= X_k$$ since $E[N_k] = 0$

$\therefore \{X_k\}$ is a martingale

QED

First: Discrete case

equilibrium condition

$\therefore$ no gain at equilibrium

Thm: If $r = \frac{D_{t+1}}{P_t}$ then $P_{t+1} = P_t$

discount rate — yield (return on asset)

Pf: Put $P_t := \sum_{n=1}^{\infty} \frac{D_{t+n}}{(1+r)^n}$ ← discounted future cash flow

$$\therefore P_{t+1} = \sum_{n=1}^{\infty} \frac{D_{t+n+1}}{(1+r)^n} = \sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^{n-1}}$$

if "back up index n" (alternatively: put $j := n+1$)

$$\therefore P_t - \frac{P_{t+1}}{1+r} = \sum_{n=1}^{\infty} \frac{D_{t+n}}{(1+r)^n} - \frac{1}{1+r} \sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{D_{t+n}}{(1+r)^n} - \sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n}$$

$$= \left(\frac{D_{t+1}}{1+r} + \cancel{\sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n}} \right) - \cancel{\sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n}}$$

$$= \frac{D_{t+1}}{1+r}$$

$$\therefore \boxed{P_{t+1} = (1+r) P_t - D_{t+1}}$$

Suppose equilibrium condition $r = \frac{D_{t+1}}{P_t}$ $\therefore (1+r) = 1 + \frac{D_{t+1}}{P_t}$

$$\therefore P_{t+1} = (1+r) P_t - D_{t+1} = \left(1 + \frac{D_{t+1}}{P_t}\right) P_t - D_{t+1}$$

$$= P_t + \cancel{D_{t+1}} - \cancel{D_{t+1}} = P_t$$

$\therefore$ At equilibrium: $P_{t+1} = P_t$

QED.

Random Case: Show that $\{P_t\}$ is a martingale

Martingale Pricing Theorem (Paul Samuelson - 1965)

If
- sigma-algebras $I_t \subset I_{t+1}$ (filtration)
- r.v. $D_t \longleftrightarrow I_t$ (adapted)
- equilibrium condition: $r = \frac{E[D_{t+1} | I_{t}]}{P_t}$ expected return on asset.

then $E[P_{t+1} | I_t] = P_t$

$\therefore$ $\{P_t\}$ is a martingale

$\therefore$ "Can't beat an efficient market"

best estimate of tomorrow's price is today's price

Pf: r.v. $P_t \equiv$ Conditional (rational) Asset Price

$$\equiv E[P_t | I_t] \quad \text{r.v.}$$

$$= E\left[\sum_{n=1}^{\infty} \frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_t\right]$$

$$= \sum_{n=1}^{\infty} E\left[\frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_t\right]$$ assuming expectation exists

$$\therefore \quad P_{t+1} = E\left[P_{t+1} \mid I_{t+1} \right] = E\left[\sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_t \right]$$

by backing up index n

$$\therefore \quad E\left[P_{t+1} \mid I_t \right] = E\left[E\left[\sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_{t+1} \right] \,\middle|\, I_t \right]$$

total exp

crucial step

$$= E\left[\sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_t \right] \qquad \text{since } I_t \subset I_{t+1}$$

$$= E\left[\underbrace{D_{t+1} - D_{t+1}}_{=0} + \sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_t \right]$$

$$= E\left[D_{t+1} \mid I_t \right] + E\left[\sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_t \right] - E\left[D_{t+1} \mid I_t \right]$$

$$\therefore \quad \frac{1}{1+r} \cdot E\left[P_{t+1} \mid I_t \right] = \left(E\left[\frac{D_{t+1}}{1+r} \,\middle|\, I_t \right] + E\left[\sum_{n=2}^{\infty} \frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_t \right] \right) - \frac{1}{1+r} E\left[D_{t+1} \mid I_t \right]$$

$$= E\left[\sum_{n=1}^{\infty} \frac{D_{t+n}}{(1+r)^n} \,\middle|\, I_t \right] - \frac{1}{1+r} \cdot E\left[D_t \mid I_t \right]$$

lower case

$$= p_t - \frac{1}{1+r} E\left[D_{t+1} \mid I_t \right]$$

$$\therefore \quad E\left[p_{t+1} \mid I_t \right] = (1+r) p_t - E\left[D_{t+1} \mid I_t \right]$$

$$= \left(1 + \frac{E[D_{t+1} | I_t]}{P_t}\right) P_t - E[D_{t+1} | I_t]$$

equilibrium

if $r = \frac{E[D_{t+1} | I_t]}{P_t}$

$$= P_t + \cancel{E[D_{t+1} | I_t]} - \cancel{E[D_{t+1} | I_{t+1}]}$$

$$= P_t$$

$\therefore \{P_t\}$ is a martingale

QED

FINANCIAL DERIVATIVES

Idea: Martingale $\longleftrightarrow$ No Arbitrage Opportunity

(Google: "Arbitrage Theorem")

Black-Scholes (BS) value (price) of a

(European) "CALL" option

right to buy

$$\boxed{C(S,t) = S \cdot N(d_1) - Xe^{-rt} N(d_2)}$$

share price

"strike" price

normal CDF, Φ for $Z \sim N(0,1)$

$$d_2 = d_1 - \sigma\sqrt{t}$$

$$d_1 = \frac{\ln(S/X) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

time to expiration

r: risk free rate of return

- continuous compounding

For a "PUT" option

$$\boxed{P(S,t) = Xe^{-rt}N(-d_2) - SN(-d_1)}$$

Put-Call parity

$$C(S,t) + Xe^{-rt} = P(S,t) + S$$

else arbitrage possible

Myron Scholes
1997 Nobel Prize

Black, F., and Scholes, M., "The Pricing of Options and Corporate Liabilities," Journal of Political Economy, vol. 81, pp. 637-659, May-June 1973.

Two Types of Calculus

Newtonian Calculus

$$\dot{x} = f(x)$$

DE

Ito (Stochastic) Calculus

"Langevin"

$$\dot{x} = f(x) + n_t$$

r.v.

SDE

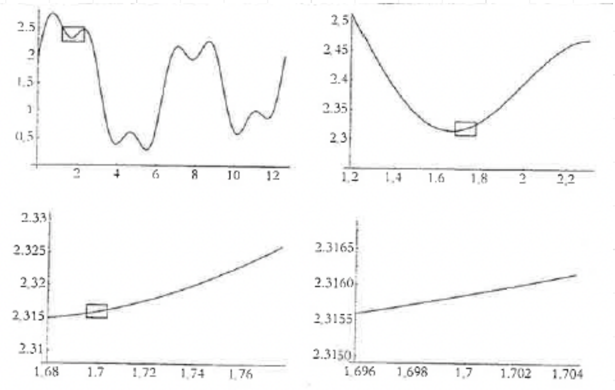
$$dx = f(x)\,dt + dB$$

Brownian Motion B_t

- $B_0 = 0$ $\quad t \geq 0$ & continuous
- $B_t \sim N(0, t)$
- Independent Increments

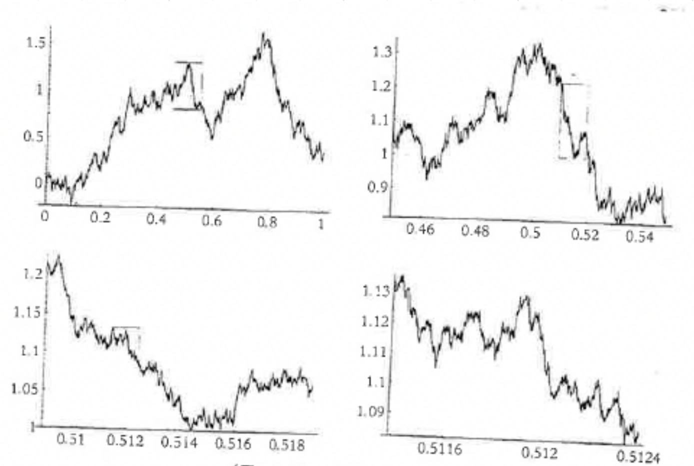
$$B_{s+t} - B_s \quad (\text{independent of } \mathcal{Q}_s)$$

$$df = \mu_t \cdot dt$$



Progressive magnification around the point 1.7

Linear approximation:



'Zooming in' on Brownian motion

$\therefore$ B_t self-similar

– "fractal"

Note: no jumps (pure diffusion)

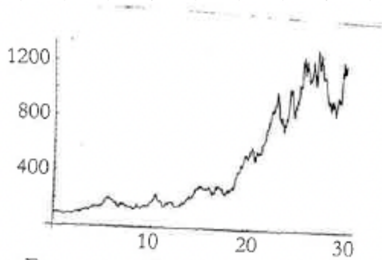
Geometric ("Exponential") BM (GBM) "with drift"

$$S_t = S_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}$$

growth term

$\sim$ Log Normal

UK FTA index, 1963–92



Exponential Brownian motion

Best fit

$\mu = 0.087$

$\sigma = 0.178$

- annual drift 8.7 %

$\therefore$ tempting to model assets $\sim$ Log Normal

$$dS_t \overset{\text{SDE}}{=} \mu S_t \, dt + \sigma S_t \, dB_t$$

"SDE"

not r.v. r.v.

$$\dot{x} = f(x) + n_t$$

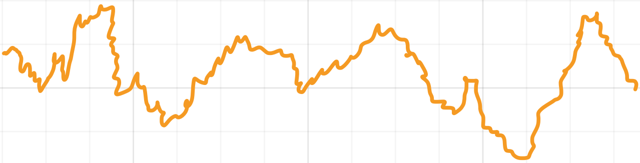
$$dx = f(x)\,dt + n_t\,dt$$

Real SDE:

$$dx = df\,dt + dB_t$$

for x a r.v.

B_t



- continuous case
- not differentiable ($\forall t$)

"kinks"

Pseudo-derivative:

$$\text{"}\frac{dB}{dt}\text{"} = n$$

Gaussian WHITE NOISE!

- Need Ito's Lemma (≅ SDE Chain-Rule)

r.v. r.v.

If $dx(t) = a(t)\,dt + b(t)\,dB_t$

& f twice-continuously differentiable

Then

$$df(x,t) = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{b^2}{2}\frac{\partial^2 f}{\partial t^2}\right)dt + b\frac{\partial f}{\partial x}\,dB_t$$

$$\left((dB)^2 = dt\right)$$ "Box Algebra"

("Box Algebra")

GBM: $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t} \sim LN.$

SDE Stock models

$$dS_t \overset{\text{GBM}}{=} \mu S_t\, dt + \sigma S_t\, dB_t$$

& $f(S_t) = \ln S_t$ — twice differentiable ($\therefore$ log-normal assumption)

$$\therefore df(S_t) = d(\ln S_t)$$

$$\overset{\text{Itô's lemma}}{=} \left(\mu \cancel{S_t} \cdot \frac{1}{\cancel{S_t}} + 0 + \frac{\sigma^2}{2} \cancel{S_t^2} \left(-\frac{1}{\cancel{S_t^2}}\right)\right) dt$$

$$+ \sigma \cancel{S_t} \cdot \frac{1}{\cancel{S_t}}\, dB_t$$

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dB_t$$

Michael Steele: Stochastic Calculus & Financial Applications (2008)

- coefficient matching (and exponential bond model)

- Leads to Black-Scholes PDE

$$\therefore\ dC = \left[\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial t^2} S^2\right] dt + \sigma S \frac{\partial C}{\partial S} dB_t$$

Leads to BS "No Arbitrage" PDE

$$\frac{\partial C}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial S^2} S^2 + rS\frac{\partial C}{\partial S} - rC = 0$$

$\therefore$ Arbitrage opportunity exists $\longleftrightarrow$ No PDE solution

---

$\therefore$ Thrm: Black-Scholes Formula for (European) call

$$C(S,t) = S \cdot N(d_1) - Xe^{-rt} N(d_2)$$

with $d_2 = d_1 - \sigma\sqrt{t}$

$$d_1 = \frac{\ln(S/X) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

Note: No μ (drift term) in BS solution
$\therefore$ No stock growth term

Put-Call parity gives BS solution for Put option.

CALL: Right to Buy
PUT: Right to Sell
stock at fixed price at T

Write options
- Covered - You own stock
- Uncovered - You don't own stock

"LEAPS" - long term options (≤ 4 years)
- crash of 1987 ?

Hedge: Buy (inherit) KO shares at \$50/share
- probate takes 1 year
- $\therefore$ Buy PUT at \$45/share

Ex:

$S =$ Share price \$47

X: Exercise ("Strike") price \$45

$t = \frac{1}{2}$ year $= \frac{183}{365}$ (European Call)

$r = 0.10$ $\quad \sigma = 0.25$ ← volatility

where? OLS, implied volatility, ARCH/GARCH

$\therefore \quad d_1 = 0.6172$

$r = 0.1$
$\sigma = 0.25$

$d_2 = 0.6172 - 0.25 \cdot \sqrt{1/2}$

$= 0.4404$

$\therefore \ c(S,t) = SN(d_1) - Xe^{-rt} N(d_2)$

$= 47 \, N(0.6172) - 45 e^{-0.05} N(0.4404)$

$= 47 (0.7315) - 45 e^{-0.05} (0.6702)$

$= \boxed{\$ 5.69}$

But if $\sigma = 0.4$

then... $c(S,t) = \boxed{\$ 7.42}$

$\therefore$ "Volatility favors the option"

$$\boxed{\frac{\partial C}{\partial \sigma} > 0}$$